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Variational Techniques on Classes of Lagrangians

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We present canonical procedures for the manipulation of whole classes of Lagrangians that share the same transformation law and functional dependence but are otherwise arbitrary in functional form, and for the derivation therefrom of generalized conserved quantities. The techniques are demonstrated on the class of scalar density Lagrangians $L = L_G + L_{EM}$, where L_G is a function of the metric and its first and second derivatives and L_{EM} is a function of the metric and a vector potential and its first derivative, which generate the Einstein-Maxwell equations (without cosmological constant). These procedures should be of interest to those studying alternate formulations of general relativity, those deriving new field theories, and others working with general of modified Lagrangians.

1. INTRODUCTION

In this paper we present canonical procedures for the manipulation of a general Lagrangian, with known functional dependence and transformation law but unknown functional form, and for the subsequent derivation of generalized conserved quantities. These techniques are thus applicable to whole classes of Lagrangians that share the same functional dependence and transformation law. In particular we may apply them to the class of scalar density Lagrangians that are functions of the metric and its first and second derivatives, the Lagrangians of general relativity. Also, while we are concerned here only with scalar density Lagrangians, it should be noted that these techniques are equally applicable to nonscalar densities provided the corresponding transformation law is known.

Historically, the study of field theories and conserved quantities has involved the determination of a single Lagrangian and an application of well-known variational techniques and Noether's theorem. However, modern theoretical investigations have become more sophisticated and the

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work presented here may prove of value to those interested in theories with several Lagrangians, in the extension of current theories through the addition of extra terms in the Lagrangian, or in the development of new field theories. But these procedures can also be applied to classical general relativity to clarify the somewhat confused state of conserved quantities in the theory, and it is this problem we address in this paper. In general relativity there are no fewer than three energy-momentum complexes in common use (Einstein, 1916; Landau and Lifshiftz, 1975; Møller, 1972) and an infinite number are known (Goldberg, 1958; Komar, 1959); none has proved wholly satisfactory. From a theoretical point of view, these complexes are now somewhat out of vogue owing to recent progress in coordinate-independent quantities (e.g., Penrose (1982)), but, as they are still in widespread use and are likely to remain so, they are not devoid of interest and will provide well-known examples for the application of the technique. Thus, in this paper we apply our canonical procedure to a Hilbert variation of a classical general relativistic scalar density Lagrangian, providing a compact derivation of a number of new and well-known momentum complexes and generalizations thereof. In a subsequent paper (Churchill, 1987) we apply the canonical procedure to a Palatini variation of a general class of Lagrangians based on that of Nissani (1985). This class includes, as a special case, the classical general relativistic Lagrangians of Lovelock (1969).

We begin, in Sections 2–5, by considering the general (Hilbert) variation of a scalar action[2]

$$S=\int L d^4x \tag{1}$$

where the scalar density Lagrangian L is a sum of electromagnetic and gravitational parts. We include the electromagnetic part in the Lagrangian and derive its contribution to the conserved quantities, despite the fact that the electromagnetic energy-momentum density can be (and usually is) obtained directly from the Einstein–Maxwell equations. Generally, the electromagnetic energy-momentum density is derived as a conserved quantity only within the framework of classical electromagnetism. However, the derivation involves manipulations to ensure symmetry. We perform the operation here, in the presence of gravity, because the symmetrization process may also be applied to the gravitational part of the combined energy-momentum complex and provides insight to the interpretations of this and intermediate complexes. In doing so we are led (for the first time, to the best of my knowledge) to a derivation of the Landau and Lifshitz pseudotensor via a variational principle.

[2]We set $c=8\pi G=1$.

In Section 2 we manipulate the transformation relations for L and its arguments in order to investigate the symmetry and tensor properties of the partial derivatives of L and deduce certain invariance relations between them. In Section 3 we derive the variation of the action and write the integrand of δS as the divergence of a vector density $h^a{}_{,a}$. At this point most authors specify the variation as an infinitesimal translation and use the properties of a specific Lagrangian to obtain a mixed energy-momentum complex $h^a{}_b$, which is then "integrated"[3] to form the superpotential. Komar (1959) derives an improved superpotential from the Hilbert Lagrangian $(\sqrt{-g})R$ in terms of an arbitrary variation δx^a. In contrast, we use the invariance relations of Section 2 to "integrate" a strongly conserved quantity h^a, which is general in the choice of both the variation and the Lagrangian. The resulting expression has several advantages. It is mathematically simpler in that most of its properties may be deduced by inspection. Also, by specifying the appropriate variation, one may generate both a mixed and contravariant energy-momentum complex and an angular momentum complex for general scalar density Lagrangians. The complexes $h^a{}_b$ and h^{ab} are derived in Section 4. The angular momentum complex h^{abc} is presented in Section 5. We show that the moment of h^{ab} constitutes only part of h^{abc}. We then use the conservation of h^{abc} to find the unaccounted-for "spin" energy contribution of the remaining part of h^{abc}, which is added to h^{ab} to produce a symmetric total energy-momentum complex H^{ab}. Finally, in Section 6 Lovelock's Lagrangian is considered and new complexes are generated along with generalizations of those in common use.

2. INVARIANCE RELATIONS

In this section we use the transformation laws of the Lagrangian and its arguments to derive symmetry and invariance relations in the functional derivatives of the Lagrangian. While these relations are not new (see, for instance, Lovelock and Rund (1975)), they seem to have found little application in the literature. We will see that we may generate a surprising amount of general information, without reference to the exact functional form of any specific Lagrangian, which may be directly applied to the "integration" of a general conserved quantity in Section 3.

We begin by considering a scalar density Lagrangian L of the form

$$L = L_{\mathrm{G}} + L_{\mathrm{EM}} \tag{2}$$

where L_{G} and L_{EM} will be considered as gravitational and electromagnetic

[3]Here and in the following we will loosely use the term "integrate" to represent the phrase "take the antidivergence."

terms with the functional dependences

$$L_G = L_G(g_{ab}, g_{ab,c}, g_{ab,cd}) \tag{3a}$$

$$L_{EM} = L_{EM}(g_{ab}, \phi_a, \phi_{a,b}) \tag{3b}$$

and L defines an action scalar S given by (1). We introduce the following notations for the derivatives of L:

$$\Lambda^{ij} \equiv \frac{\partial L}{\partial g_{ij}}, \qquad \Lambda^{ijk} \equiv \frac{\partial L}{\partial g_{ij,k}}, \qquad \Lambda^{ijkl} \equiv \frac{\partial L}{\partial g_{ij,kl}} \tag{4a}$$

$$\Phi^i \equiv \frac{\partial L}{\partial \phi_i}, \qquad \Phi^{ij} \equiv \frac{\partial L}{\partial \phi_{i,j}} \tag{4b}$$

which obey the symmetry relations:

$$\Lambda^{ij} = \Lambda^{ji} \tag{5a}$$

$$\Lambda^{ijk} = \Lambda^{jik} \tag{5b}$$

$$\Lambda^{ijkl} = \Lambda^{jikl} = \Lambda^{ijlk} \tag{5c}$$

We now consider the transformation laws for $\bar{L}(\bar{g}_{ij}, \bar{g}_{ij,k}, \bar{g}_{ij,kl}, \bar{\phi}_i, \bar{\phi}_{i,j})$ and its arguments. Differentiating these relations with respect to the arguments of L yields transformation laws for the functional derivatives of L, from which we see that Λ^{abcd} and Φ^{ab} transform as tensor densities, while Λ^{ab}, Λ^{abc}, and Φ^a do not [as a conseuqence of (6c) below, Φ^a is, in fact, tensorial].

Finally, we derive the invariance relations. The procedure consists of two steps: we differentiate the transformation laws of $\bar{L}$ and its arguments with respect to the coordinate transformation $\partial x^p/\partial \bar{x}^q$ and its derivatives, whereupon we consider the particular case of the identity transformation. After some manipulation we find

$$\Lambda^{iqrs} + \Lambda^{irsq} + \Lambda^{isqr} = 0 \tag{6a}$$

$$\Lambda^{ijkl} = \Lambda^{klij} \tag{6b}$$

$$\Phi^{qr} + \Phi^{rq} = 0 \tag{6c}$$

$$\Lambda^{pqr} + \Lambda^{prq} = 2\Lambda^{ijqr}\Gamma^p{}_{ij} \tag{6d}$$

$$\Lambda^{qrp} = \Gamma^q{}_{ij}\Lambda^{ijrp} + \Gamma^r{}_{ij}\Lambda^{ijpq} - \Gamma^p{}_{ij}\Lambda^{ijqr} \tag{6e}$$

$$\Lambda^{abc} - \Gamma^{c}_{ij}\Lambda^{abij} = 2(\Lambda^{abcd}{}_{,d} - \Lambda^{abcd}{}_{;d}) \tag{6f}$$

$$2\,{}_{\mathrm{EM}}\Lambda^{iq}g_{ip} + \Phi^{q}\phi_{p} + \Phi^{qj}F_{jp} = \delta^{q}{}_{p}L_{\mathrm{EM}} \tag{6g}$$

$$2\,{}_{\mathrm{G}}\Lambda^{iq}g_{ip} + 2\Lambda^{iqk}g_{ip,k} + \Lambda^{ijq}g_{ij,p} + 2\Lambda^{iqkl}g_{ip,kl} + 2\Lambda^{ijkq}g_{ij,kp} = \delta^{q}{}_{p}L_{\mathrm{G}} \tag{6h}$$

where

$$F_{jp} \equiv \phi_{p,j} - \phi_{j,p} \tag{6i}$$

and ${}_{\mathrm{EM}}\Lambda^{iq}$ and ${}_{\mathrm{G}}\Lambda^{iq}$ are, respectively, the electromagnetic and gravitational contributions to Λ^{iq} [with an appropriate Lagrangian ${}_{\mathrm{EM}}\Lambda^{iq}$ is just $(\sqrt{-g})\,T^{iq}$, where T^{iq} is the usual symmetric stress-energy tensor of the electromagnetic field]. These relations are completely general in that they hold for any scalar density Lagrangian (2) with functional dependences (3).

3. THE VARIATION OF THE ACTION AND THE "INTEGRATION" OF THE CONSERVED COMPLEX h^a

We begin this section with a general variation of the action (1). Rather than separately treating each of the terms in (2) to individual variations, we subject the total Lagrangian to a simultaneous variation of the coordinates x^a and potentials ϕ_a and g_{ab}. This permits the representation of the variation as a single infinitesimal coordinate transformation, which, in concert with the invariance relations derived in the previous section, permits the "integration" of a conserved quantity h^a in terms of both an arbitrary variation and scalar density Lagrangian. The resulting expression is a generalization of Komar's complex.

The general variation of the coordinates and potentials of the action (1) results in the expression (Barut, 1965)

$$\delta S = \int_{R} \delta L\, d^4x + \int_{\partial R} L\,\delta x^a\, dS_a \tag{7}$$

where ∂R denotes the boundary of the region R. If we require that δx^a and its first and second derivatives vanish on ∂R, then the invariance of the action yields the Euler-Lagrange equations

$$E^{ij} \equiv -\Lambda^{ij} + \Lambda^{ijk}{}_{,k} - \Lambda^{ijkl}{}_{,lk} = 0 \tag{8a}$$

$$E^{i} \equiv -\Phi^{i} + \Phi^{ij}{}_{,j} = 0 \tag{8b}$$

which, with an appropriate Lagrangian, reduce to the Einstein-Maxwell equations. Note that both E^i and E^{ij} are tensor densities.

We now write

$$\delta S = 2\int h^{a}{}_{,a}\, d^4x \tag{9}$$

and define the vector $\chi^a \equiv \delta x^a$. In the following we derive a strong conservation law by "integrating" the quantity h^a without making reference to equations (8).

Under the infinitesimal transformation $\bar{x}^a = x^a + \chi^a$, the variations of the potentials may be written

$$\delta g_{ij} = -(\chi_{i;j} + \chi_{j;i}) \tag{10a}$$

$$\delta \phi_i = -(\phi_a \chi^a)_{,i} - F_{ai}\chi^a \tag{10b}$$

Taking the covariant derivative of equations (6g) and (6h) yields

$$\begin{aligned} L_{\mathrm{EM};p} &= {}_{\mathrm{EM}}\Lambda^{ij} g_{ij;p} + \Phi^i \phi_{i;p} + \Phi^{ij}\phi_{i,j;p} \\ &= (2\,{}_{\mathrm{EM}}\Lambda^{iq} g_{ip} + \Phi^q \phi_p + \Phi^{qj} F_{jp})_{;q} \end{aligned} \tag{11a}$$

$$\begin{aligned} L_{\mathrm{G};p} &= {}_{\mathrm{G}}\Lambda^{ij} g_{ij;p} + \Lambda^{ijk} g_{ij,k;p} + \Lambda^{ijkl} g_{ij,kl;p} \\ &= (2\,{}_{\mathrm{G}}\Lambda^{iq} g_{ip} + 2\Lambda^{iqk} g_{ip,k} + \Lambda^{ijq} g_{ij,p} \\ &\quad + 2\Lambda^{iqkl} g_{ip,kl} + 2\Lambda^{ijkq} g_{ij,kp})_{;q} \end{aligned} \tag{11b}$$

Substitution of these into the expression for δL gives

$$\begin{aligned} \delta L &= (2E^{ia}\chi_i + E^a \phi_i \chi^i)_{;a} \\ &\quad + [(\Lambda^{ijk} - \Lambda^{ijkl}{}_{,l})\delta g_{ij} + \Lambda^{ijkl}\delta g_{ij,l} + \Phi^{ik}\delta\phi_i]_{,k} \end{aligned} \tag{12}$$

Thus we have the strongly conserved quantity

$$\begin{aligned} h^a &= E^{ia}\chi_i + \tfrac{1}{2}E^a \phi_i \chi^i + \tfrac{1}{2}(\Lambda^{ija} - \Lambda^{ijal}{}_{,l})\,\delta g_{ij} \\ &\quad + \tfrac{1}{2}\Lambda^{ijal}\delta g_{ij,l} + \tfrac{1}{2}\Phi^{ia}\delta\phi_i + \tfrac{1}{2}L\chi^a \end{aligned} \tag{13}$$

After substituting for E^{ia}, E^i, δg_{ij}, $\delta g_{ij,l}$, and $\delta\phi_i$, h^a may be "integrated" once more. The calculation yields

$$h^a = [\tfrac{1}{2}\Phi^{ak}\phi_n \chi^n - \Lambda^{ijak}\chi_{j;i} + 2\chi_j(\Lambda^{kjal}{}_{;l} + \tfrac{1}{2}\Gamma^a{}_{il}\Lambda^{ilkj})]_{,k} \tag{14}$$

This expression, which is somewhat unwieldy, may be considerably simplified by the introduction of the useful quantity

$$\psi^{abcd} \equiv \tfrac{1}{3}(\Lambda^{abcd} - \Lambda^{adcb}) \tag{15}$$

From this definition and equations (5c), (6a), and (6b), we may show that ψ^{abcd} has the following properties:

$$\Lambda^{abcd} = \psi^{abcd} + \psi^{abdc} \tag{16a}$$

$$\psi^{abcd} = -\psi^{cbad} = -\psi^{adcb} = \psi^{badc} = \psi^{cdab} \tag{16b}$$

$$\psi^{abcd} + \psi^{adbc} + \psi^{acdb} = 0 \tag{16c}$$

Then, after the substitution of (16a) into (14), a short calculation yields

$$\begin{aligned} h^a &= (\tfrac{1}{2}\Phi^{ak}\phi_n\chi^n + \psi^{abkl}\chi_{b;l} - 2\psi^{abkl}{}_{;l}\chi_b)_{;k} \\ &= (\tfrac{1}{2}\Phi^{ak}\phi_n\chi^n + \psi^{abkl}\chi_{b,l} - 2\psi^{abkl}{}_{;l}\chi_b)_{,k} \end{aligned} \tag{17}$$

It is readily apparent from (17) that h^a is a vector density with vanishing divergence (if we wish, we may define h^a as the divergence of an antisymmetric superpotential, itself a tensor density). Thus, (17) constitutes a strong conservation law, general in the Lagrangian (2), which generates a conserved quantity for any specified variation χ^a. This new expression constitutes a generalization of Komar's complex.

4. DERIVATION OF CONSERVED QUANTITIES FROM h^a

In order to generate physically interesting conserved quantities from the complex h^a we consider the variation of the previous section as given by an infinitesimal transformation defined in terms of an arbitrary set of independent parameters $\delta k^A{}_B$, where the capitals represent sets of indices. That is, we write

$$\chi^a = f^a{}_A{}^B\,\delta k^A{}_B \tag{18}$$

where $f^a{}_A{}^B$ is some function of the coordinates and potentials and the $\delta k^A{}_B$, which are just the infinitesimal generators of the group whose "motion" represents the symmetry of the spacetime, are to be considered as arbitrary but predetermined, and thus constant with respect to the coordinates.

As in Hamilton–Jacobi theory we may then write

$$\delta S/2\,\delta k^A{}_B = \int h^a{}_A{}^B\,dS_a \tag{19}$$

(note that, up to a linear transformation, this fixes the coordinates.) Now, by specifying the appropriate infinitesimal generators $\delta k^A{}_B$, we may write the "momenta" P_s, P^t and the "angular momentum" J^{ts} as surface integrals of the energy-momentum complexes $h^a{}_s$, h^{at} and the angular momentum complex h^{ats}. These $\delta k^A{}_B$ derive from the infinitesimal vectors

$$\xi^s = \gamma^s{}_n x^n + \zeta^s \tag{20a}$$

$$\xi_t = \gamma_{tn} x^n + \zeta_t \tag{20b}$$

where γ_{tn} is antisymmetric. But ξ^s, ξ_t are just the Killing vectors of Minkowski space and do not generally represent true symmetries of the spacetime. Only by integrating near infinity on an asymptotically flat spacetime may we be sure of obtaining valid results. Thus the motivation for the term "complex"; these objects are not true momentum densities and, in general, will not exhibit the corresponding local properties.

Letting

$$\chi^n = (\sqrt{-g})^{n-1}\delta^n{}_s\,\delta k^s \tag{21}$$

we obtain from equation (17) the mixed complex

$$\begin{aligned} {}_{(n)}h^a{}_s = \{&\tfrac{1}{2}(\sqrt{-g})^{n-1}\Phi^{ak}\phi_s + [(\sqrt{-g})^{n-1}g_{bs}]_{,l}\psi^{abkl} \\ &-2(\sqrt{-g})^{n-1}g_{bs}\psi^{abkl}{}_{;l}\}_{,k} \end{aligned} \tag{22}$$

and, if we let

$$\chi^n = (\sqrt{-g})^{n-1}g^{nt}\delta k_t \tag{23}$$

we in turn generate the contravariant complex

$${}_{(n)}h^{at} = [\tfrac{1}{2}(\sqrt{-g})^{n-1}\Phi^{ak}\phi^t + (\sqrt{-g})^{n-1}{}_{,l}\psi^{atkl} - 2(\sqrt{-g})^{n-1}\psi^{atkl}{}_{;l}]_{,k} \tag{24}$$

where we have written $(\sqrt{-g})^{n-1}$ to denote $\sqrt{-g}$ to the power $n-1$ and ${}_{(n)}h^{at}$ to denote the weight n complex.

With the introduction of a general relativistic Lagrangian, objects derived from equations (22) and (24) will, under the appropriate coordinate conditions, correctly give the global values for energy and momentum. However, equation (24) is not symmetric and thus the moment of ${}_{(n)}h^{at}$ does not define a conserved angular momentum complex. Further, both ${}_{(n)}h^a{}_s$ and ${}_{(n)}h^{at}$ contain a bothersome term in Φ^{ak}. With the introduction of the field equations (8a) into these expressions we find those parts containing electromagnetic terms to be

$${}_{(n)}t_{\mathrm{EM}}{}^a{}_s = (\sqrt{-g})^{n-1}\,{}_{\mathrm{EM}}\Lambda^{an}g_{ns} + [\tfrac{1}{2}(\sqrt{-g})^{n-1}\Phi^{ak}\phi_s]_{,k} \tag{25a}$$

$${}_{(n)}t_{\mathrm{EM}}{}^{at} = (\sqrt{-g})^{n-1}\,{}_{\mathrm{EM}}\Lambda^{at} + [\tfrac{1}{2}(\sqrt{-g})^{n-1}\Phi^{ak}\phi^t]_{,k} \tag{25b}$$

[recall that with an appropriate Lagrangian ${}_{\mathrm{EM}}\Lambda^{at} = (\sqrt{-g})\,T^{at}$], from which we see that the final terms must be eliminated if we are to obtain correctly the usual electromagnetic stress-energy tensor. These terms may be discarded ad hoc since they are divergenceless. However, it is instructive to seek a more illustrative basis for their elimination, which may lend itself to some physical interpretation. An appropriate procedure is suggested by appealing to electromagnetic field theory.

In the absence of a gravitational field, both equations (25a) and (25b) represent the same object

$${}_{(n)}t_{\mathrm{EM}}{}^{at} = {}_{\mathrm{EM}}\Lambda^{at} + (\tfrac{1}{2}\Phi^{ak}\phi^t)_{,k} \tag{26}$$

which, for the usual Maxwell Lagrangian, reduces to the nonsymmetric stress-energy tensor (Landau and Lifshitz (1975)),

$$t_{\mathrm{EM}}{}^{at} = \tfrac{1}{2}(-4\phi_{i,j}\eta^{jt}F^{ai} + \eta^{at}F_{ij}F^{ij}) \tag{27}$$

where η^{at} is the Minkowski metric. Equation (27) can be symmetrized, producing the normal electromagnetic stress-energy tensor, through the addition of a divergenceless term obtained via the conservation of the angular momentum density. This term is interpreted as the as yet uncounted energy-momentum contribution of that part of the angular momentum density not represented by the moment of (27). The presence of this anomalous electromagnetic term in our calculation in the presence of gravitation seems to imply a similarly uncounted gravitational energy-momentum contribution, which we may include via a symmetrization of the whole gravitational-electromagnetic complex. We perform this operation in the following section.

5. DERIVATION OF THE ANGULAR MOMENTUM COMPLEX AND THE SYMMETRIZATION OF ${}_{(n)}h^{at}$

The angular momentum complex is generated from an infinitesimal rotation. Thus, we set

$$\chi^n = (\sqrt{-g})^{n-1}(g^{ns}x^t - g^{nt}x^s)\,\delta k_{ts} \tag{28}$$

in which case (19) and (17) define the object

$$\begin{aligned}{}_{(n)}h^{ats} &= {}_{(n)}h^{as}x^t - {}_{(n)}h^{at}x^s + [(\sqrt{-g})^{n-1}(\psi^{askt} - \psi^{atks})]_{,k}\\ &\quad + [\tfrac{1}{2}(\sqrt{-g})^{n-1}(\Phi^{at}\phi^s - \Phi^{as}\phi^t) + (\sqrt{-g})^{n-1}{}_{,l}(\psi^{astl} - \psi^{atsl})\\ &\quad - 2(\sqrt{-g})^{n-1}(\psi^{astl} - \psi^{atsl})_{;l}]\\ &= {}_{(n)}M^{ats} + {}_{(n)}S^{ats}\end{aligned} \tag{29}$$

where ${}_{(n)}M^{ats}$ is the moment of the complex ${}_{(n)}h^{at}$, and ${}_{(n)}S^{ats}$ represents an intrinsic field momentum (in quantum mechanics this term is used to derive spin). The energy inherent in the ${}_{(n)}S^{ats}$ portion of ${}_{(n)}h^{ats}$ has not yet been accounted for; thus, we add an additional "spin" term ${}_{(n)}s^{at}$ to ${}_{(n)}h^{at}$ to obtain a total energy-momentum complex

$${}_{(n)}H^{at} = {}_{(n)}h^{at} + {}_{(n)}s^{at} \tag{30}$$

The expression for ${}_{(n)}s^{at}$ is derived through the following set procedure (see Corson, 1955).

We begin by writing the conservation law for the angular momentum complex

$$h^{ats}{}_{,a} = h^{ts} - h^{st} + S^{ats}{}_{,a} \tag{31a}$$

$$\equiv h^{ts} - h^{st} + (\mu^{tsa} - \mu^{sta})_{,a} \tag{31b}$$

so that $h^{ts} + \mu^{tsa}{}_{,a}$ defines a symmetric object. Thus, if we let

$$s^{ts} = \mu^{tsa}{}_{,a} \tag{32}$$

$H^{at}{}_{,a}$ will vanish if and only if $\mu^{atb}{}_{,ba}$ vanishes; that is, if and only if

$$\mu^{atb} = -\mu^{bta} \tag{33}$$

From equations (31)-(33) we may infer

$$s^{ts} = \tfrac{1}{2}(S^{ats} + S^{sta} + S^{tsa})_{,a} \tag{34}$$

which, after substitution of S^{ats} from (29), becomes

$$_{(n)}s^{ts} = -2[(\sqrt{-g})^{n-1}\psi^{tsak}]_{,ka} - {}_{(n)}h^{ts} \tag{35}$$

Substitution of (35) into (30) yields the total energy-momentum complex

$$\begin{aligned} _{(n)}H^{ts} &= -2[(\sqrt{-g})^{n-1}\psi^{tsak}]_{,ka} \\ &= -[(\sqrt{-g})^{n-1}\Lambda^{tsak}]_{,ka} \end{aligned} \tag{36}$$

Reference to (16b) shows that (36) is indeed symmetric and vanishes under a divergence of either index.

6. PRESENTATION OF PARTICULAR ENERGY-MOMENTUM COMPLEXES

The formalism presented thus far has been sufficient to generate the field equations, the general conserved quantity h^a, the energy-momentum complexes $_{(n)}h^a{}_s$ and $_{(n)}h^{at}$, the angular momentum complex $_{(n)}h^{ats}$, and the total energy-momentum complex $_{(n)}H^{ts}$; all without reference to any particular Lagrangian. We will see that these quantities suffice to generate and generalize virtually all energy-momentum complexes currently known.[4] We will now present the scalar density results.

The most general scalar density Lagrangian that generates the Einstein–Maxwell equations without the cosmological term is (Lovelock, 1969, 1974)

$$L = L_{\mathrm{H}} + L_{\mathrm{M}} + \alpha L_\alpha + \beta L_\beta + \gamma L_\gamma \tag{37}$$

where

$$L_{\mathrm{H}} = (\sqrt{-g})R \tag{38a}$$

$$L_\alpha = \varepsilon^{ijkl}R^{ab}{}_{ij}R_{abkl} \tag{38b}$$

$$L_\beta = (\sqrt{-g})[RR - 4R_{ij}R^{ij} + R^{ij}{}_{kl}R^{kl}{}_{ij}] \tag{38c}$$

$$L_{\mathrm{M}} = (\sqrt{-g})F^{ij}F_{ij} \tag{38d}$$

$$L_\gamma = \varepsilon^{ijkl}F_{ij}F_{kl} \tag{38e}$$

and α, β, and γ are arbitrary constants.

[4]The notable exceptions are the complexes of Einstein (1916) and Weinberg (1972). Weinberg's complex does not lend itself to derivation via a variational principle. However, our technique may be applied to the Einstein Lagrangian to derive Einstein's complex, although the nonscalar nature of this Lagrangian complicates the analysis and only the mixed weight-one complex $_{(1)}h^a{}_s$ (Einstein's) exists.

The resulting quantities relevant in the calculation of the complexes are

$$ {}_{\mathrm{H}}\psi^{abcd} = -\tfrac{1}{2}(\sqrt{-g})(g^{ab}g^{cd} - g^{ad}g^{bc}) \tag{39a} $$

$$ \begin{aligned} {}_{\alpha}\psi^{abcd} &= -\tfrac{1}{3}(2\varepsilon^{ijac}R^{bd}{}_{ij} + 2\varepsilon^{ijbd}R^{ac}{}_{ij} + \varepsilon^{ijad}R^{bc}{}_{ij} \\ &\quad + \varepsilon^{ijbc}R^{ad}{}_{ij} + \varepsilon^{ijab}R^{cd}{}_{ij} + \varepsilon^{ijcd}R^{ab}{}_{ij}) \end{aligned} \tag{39b} $$

$$ \begin{aligned} {}_{\beta}\psi^{abcd} &= -2(\sqrt{-g})[R^{acbd} - (g^{ab}R^{cd} + g^{cd}R^{ab} - g^{ad}R^{bc} - g^{bc}R^{ad}) \\ &\quad + \tfrac{1}{2}R(g^{ab}g^{cd} - g^{ad}g^{bc})] \end{aligned} \tag{39c} $$

$$ {}_{\mathrm{M}}\Phi^{ab} = -4(\sqrt{-g})F^{ab} \tag{39d} $$

$$ {}_{\gamma}\Phi^{ab} = 4\varepsilon^{abij}F_{ij} \tag{39e} $$

Rewriting (17), (22), (24), and (36) in terms of the quantities (39) yields

$$ \begin{aligned} h^{a} &= [\tfrac{1}{2}({}_{\mathrm{M}}\Phi^{ak} + \gamma\,{}_{\gamma}\Phi^{ak})\phi_{n}\chi^{n} + ({}_{\mathrm{H}}\psi^{abkl} + \alpha\,{}_{\alpha}\psi^{abkl} \\ &\quad + \beta\,{}_{\beta}\psi^{abkl})\chi_{b,l} - 2\alpha\,{}_{\alpha}\psi^{abkl}{}_{;l}\chi_{b}]_{,k} \end{aligned} \tag{40a} $$

$$ \begin{aligned} &= (2[\gamma\varepsilon^{akij}F_{ij} - (\sqrt{-g})F^{ak}]\phi_{n}\chi^{n} - \{\tfrac{1}{2}(\sqrt{-g})(g^{ab}g^{kl} - g^{al}g^{bk}) \\ &\quad + \tfrac{1}{3}\alpha(2\varepsilon^{ijak}R^{bl}{}_{ij} + 2\varepsilon^{ijbl}R^{ak}{}_{ij} + \varepsilon^{ijal}R^{bk}{}_{ij} \\ &\quad + \varepsilon^{ijbk}R^{al}{}_{ij} + \varepsilon^{ijab}R^{kl}{}_{ij} + \varepsilon^{ijkl}R^{ab}{}_{ij}) \\ &\quad + 2\beta(\sqrt{-g})[R^{akbl} - (g^{ab}R^{kl} + g^{kl}R^{ab} - g^{al}R^{bk} \\ &\quad - g^{bk}R^{al}) + \tfrac{1}{2}R(g^{ab}g^{kl} - g^{al}g^{bk})]\}\chi_{b,l} \\ &\quad + \tfrac{4}{3}\alpha(2\varepsilon^{ijak}R^{b}{}_{i;j} + \varepsilon^{ijbk}R^{a}{}_{i;j} + \varepsilon^{ijab}R^{k}{}_{i;j})\chi_{b})_{,k} \end{aligned} \tag{40b} $$

$$ \begin{aligned} {}_{(n)}h^{a}{}_{s} &= [\tfrac{1}{2}(\sqrt{-g})^{n-1}({}_{\mathrm{M}}\Phi^{ak} + \gamma\,{}_{\gamma}\Phi^{ak})\phi_{s} \\ &\quad + [(\sqrt{-g})^{n-1}g_{bs}]_{,l}({}_{\mathrm{H}}\psi^{abkl} + \alpha\,{}_{\alpha}\psi^{abkl} + \beta\,{}_{\beta}\psi^{abkl}) \\ &\quad - 2\alpha(\sqrt{-g})^{n-1}g_{bs}\,{}_{\alpha}\psi^{abkl}{}_{;l}] \end{aligned} \tag{41a} $$

$$ \begin{aligned} &= (2(\sqrt{-g})^{n-1}[\gamma\varepsilon^{akij}F_{ij} - (\sqrt{-g})F^{ak}]\phi_{s} \\ &\quad - [(\sqrt{-g})^{n-1}g_{bs}]_{,l}\{\tfrac{1}{2}(\sqrt{-g})(g^{ab}g^{kl} - g^{al}g^{bk}) \\ &\quad + \tfrac{1}{3}\alpha(2\varepsilon^{ijak}R^{bl}{}_{ij} + 2\varepsilon^{ijbl}R^{ak}{}_{ij} + \varepsilon^{ijal}R^{bk}{}_{ij} \\ &\quad + \varepsilon^{ijbk}R^{al}{}_{ij} + \varepsilon^{ijab}R^{kl}{}_{ij} + \varepsilon^{ijkl}R^{ab}{}_{ij}) \end{aligned} $$

$$+2\beta(\sqrt{-g})[R^{akbl}-(g^{ab}R^{kl}+g^{kl}R^{ab}-g^{al}R^{bk}-g^{bk}R^{al})$$
$$+\tfrac{1}{2}R(g^{ab}g^{kl}-g^{al}g^{bk})]\}$$
$$+\tfrac{4}{3}\alpha(\sqrt{-g})^{n-1}(2\varepsilon^{ijak}R_{si;j}+g_{bs}\varepsilon^{ijbk}R^{a}{}_{i;j}+g_{bs}\varepsilon^{ijab}R^{k}{}_{i;j}))_{,k} \tag{41b}$$

$$_{(n)}h^{at}=[\tfrac{1}{2}(\sqrt{-g})^{n-1}({}_{\mathrm{M}}\Phi^{ak}+\gamma\,{}_{\gamma}\Phi^{ak})\phi^{t}+(\sqrt{-g})^{n-1}{}_{,l}({}_{\mathrm{H}}\psi^{atkl}$$
$$+\alpha\,{}_{\alpha}\psi^{atkl}+\beta\,{}_{\beta}\psi^{atkl})-2\alpha(\sqrt{-g})^{n-1}\,{}_{\alpha}\psi^{atkl}{}_{;l}]_{,k} \tag{42a}$$
$$=(2(\sqrt{-g})^{n-1}[\gamma\varepsilon^{akij}F_{ij}-(\sqrt{-g})F^{ak}]\phi^{t}$$
$$-(\sqrt{-g})^{n-1}{}_{,l}\{\tfrac{1}{2}(\sqrt{-g})(g^{at}g^{kl}-g^{al}g^{tk})$$
$$+\tfrac{1}{3}\alpha(2\varepsilon^{ijak}R^{tl}{}_{ij}+2\varepsilon^{ijtl}R^{ak}{}_{ij}+\varepsilon^{ijal}R^{tk}{}_{ij}$$
$$+\varepsilon^{ijtk}R^{al}{}_{ij}+\varepsilon^{ijat}R^{kl}{}_{ij}+\varepsilon^{ijkl}R^{at}{}_{ij})$$
$$+2\beta(\sqrt{-g})[R^{aktl}-(g^{at}R^{kl}+g^{kl}R^{at}-g^{al}R^{tk}-g^{tk}R^{al})$$
$$+\tfrac{1}{2}R(g^{at}g^{kl}-g^{al}g^{tk})]\}$$
$$+\tfrac{4}{3}\alpha(\sqrt{-g})^{n-1}(2\varepsilon^{ijak}R^{t}{}_{i;j}+\varepsilon^{ijtk}R^{a}{}_{i;j}+\varepsilon^{ijat}R^{k}{}_{i;j}))_{,k} \tag{42b}$$

$$_{(n)}H^{at}=-2[(\sqrt{-g})^{n-1}({}_{\mathrm{H}}\psi^{atkl}+\alpha\,{}_{\alpha}\psi^{atkl}+\beta\,{}_{\beta}\psi^{atkl})]_{,lk} \tag{43a}$$
$$=\{(\sqrt{-g})^{n}(g^{at}g^{kl}-g^{al}g^{tk})+2\alpha(\sqrt{-g})^{n-1}(\varepsilon^{ijak}R^{tl}{}_{ij}+\varepsilon^{ijtl}R^{ak}{}_{ij})$$
$$+4\beta(\sqrt{-g})^{n}[R^{aktl}-(g^{at}R^{kl}+g^{kl}R^{at}-g^{al}R^{tk}$$
$$-g^{tk}R^{al})+\tfrac{1}{2}R(g^{at}g^{kl}-g^{al}g^{tk})]\}_{,lk} \tag{43b}$$

Note that neither L_α, L_β, nor L_γ has any effect on the Einstein-Maxwell equations. L_γ does not contribute to the usual electromagnetic stress energy tensor [see (43)] and hence is rarely encountered. L_α and L_β both contribute to the complexes (41)-(43) and other objects derived from (40) and neither contribution vanishes with the connection, but at the same time they are nonzero in vacuum. The presence of the Riemann tensor in the α and β terms is also worthy of note. These may prove of interest in studies of gravitational radiation.

Having presented the general formulas, we now derive previously encountered energy-momentum complexes. Other than the weight-one α and β terms in $_{(n)}h^{a}{}_{s}$ [which can be found in Goenner and Kohler (1974, 1975)], these all require $\alpha=\beta=\gamma=0$.

Equations (40) now read

$$h^{a}=(\tfrac{1}{2}\,{}_{\mathrm{M}}\Phi^{ak}\phi_{n}\chi^{n}+{}_{\mathrm{H}}\psi^{abkl}\chi_{b,l})_{,k} \tag{44a}$$
$$=[\tfrac{1}{2}(\sqrt{-g})(-4F^{ak}\phi_{n}\chi^{n}+\chi^{k;a}-\chi^{a;k})]_{,k} \tag{44b}$$

which, aside from the electromagnetic term, is Komar's complex. Equations (41)-(43) become

$$
\begin{aligned}
{}_{(n)}h^{a}{}_{s} &= \{\tfrac{1}{2}(\sqrt{-g})^{n-1}\,{}_{\mathrm{M}}\Phi^{ak}\phi_{s} + [(\sqrt{-g})^{n-1}g_{bs}]_{,l}\,{}_{\mathrm{H}}\psi^{abkl}\}_{,k} && (45\mathrm{a})\\
&= \tfrac{1}{2}[-4(\sqrt{-g})^{n}F^{ak}\phi_{s} + (\sqrt{-g})^{n}g^{am}g^{kn}(g_{ms,n}-g_{ns,m}) \\
&\quad + (\sqrt{-g})^{n-1}{}_{,n}(\sqrt{-g})(\delta^{a}{}_{s}g^{nk}-\delta^{k}{}_{s}g^{an})]_{,k} && (45\mathrm{b})\\
{}_{(n)}h^{at} &= [\tfrac{1}{2}(\sqrt{-g})^{n-1}\,{}_{\mathrm{M}}\Phi^{ak}\phi^{t} + (\sqrt{-g})^{n-1}{}_{,l}\,{}_{\mathrm{H}}\psi^{abkl}]_{,k} && (46\mathrm{a})\\
&= -\tfrac{1}{2}[4(\sqrt{-g})^{n}F^{ak}\phi^{t} + (\sqrt{-g})^{n-1}{}_{,l}(\sqrt{-g})(g^{at}g^{kl}-g^{al}g^{tk})]_{,k} && (46\mathrm{b})\\
{}_{(n)}H^{at} &= -2[(\sqrt{-g})^{n-1}\,{}_{\mathrm{H}}\psi^{atkl}]_{,lk} && (47\mathrm{a})\\
&= [(\sqrt{-g})^{n}(g^{at}g^{kl}-g^{al}g^{tk})]_{,lk} && (47\mathrm{b})
\end{aligned}
$$

Neglecting the electromagnetic terms, we see that (45) is a generalization of the ($n=1$) Møller (1972) complex to arbitrary weight. Equation (46) is the generalization to arbitrary weight of a complex briefly considered by Lorentz and Levi-Civita and later rejected by Einstein (see Pauli, 1958). For $n=1$ the nonelectromagnetic terms vanish. Thus, introducing the field equations, we may write the gravitational part of ${}_{(1)}h^{at}$ in the form

$$ {}_{(1)}t^{at}_{G} = (\sqrt{-g})G^{at} \tag{48}$$

where G^{at} is the Einstein tensor. This complex was considered unsuitable because it permits the existence of nonempty spacetimes with zero total energy. Equation (47) gives the infinite family of complexes obtained by Goldberg, as generalizations of the Landau and Lifshitz complex, which is given by (47) with $n=2$. To the best of my knowledge, this is the first time the Landau and Lifshitz complex has been derived via a variational principle.

The complex ${}_{(n)}H^{at}$ is of interest on anther note. Setting $n=1$ and substituting (36) into (8a), we have (dropping the weight subscript)

$$ H^{ij} = \Lambda^{ij} - \Lambda^{ijk}{}_{,k} \tag{49}$$

This is just the expression given by Landau and Lifshitz (1975) for the total energy-momentum density of a nongravitational field (note that our definition $c=8\pi G=1$ differs from that of Landau and Lifshitz). Landau and Lifshitz claim that the relation does not hold for gravitation, but part of their argument depends on interpreting the term on the left as the total nongravitational energy (the stress-energy tensor). With the appropriate interpretation of H^{ij} as a total energy-momentum complex this relation may also be seen to hold (in a global sense at least) in the presence of gravity.

7. CONCLUSIONS

In this paper we have presented a canonical procedure for the derivation of conserved quantities for whole classes of Lagrangians that share the same transformation law and functional dependence but are otherwise arbitrary in functional form. The general technique was illustrated by its application to the Hilbert variation of a class of scalar density Lagrangians for the general relativistic gravitational and electromagnetic fields. Invariance relations, derived from the transformation relations of the Lagrangian and its arguments, were applied to permit the "integration" of a strongly conserved complex that is general in both the Lagrangian and the variation. Specification of appropriate variations led to the generation of mixed and contravariant energy-momentum complexes $h^a{}_s$ and h^{at} and an angular momentum complex h^{ats}, all general in the Lagrangian. Conservation of the angular momentum complex then permitted the construction of a symmetric complex H^{at}, also general in the Lagrangian. Specification of particular Lagrangians resulted in the presentation of new energy-momentum complexes and the presentation and generalization of virtually all such complexes previously known. This canonical procedure should be useful to researchers studying alternate formulations of general relativity, those deriving new field theories, and others working with general or modified Lagrangians. Finally, as to the future, it should be noted that while the complexes presented here have all been coordinate-dependent, these techniques are equally applicable to spinor or twistor formulations of general relativity, wherein lie hopes of generating coordinate-independent conserved quantities for general Lagrangians. It is possible that such work may lead to a variational derivation of quasilocal momentum densities with properties like those of Penrose's mass.

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